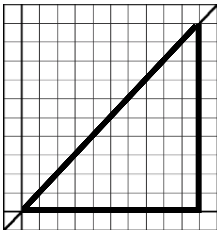
Calculus 140, section 5.1 Preparation for the Definite Integral

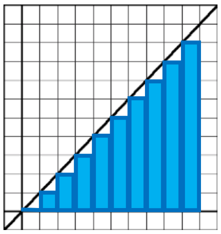
notes by Tim Pilachowski



Consider the function $f(x) = x$ on the interval [0, 10]. With the x-axis (the horizontal line $y = 0$) and the vertical line $x = 10$, f forms a triangle. We could find the area of the triangle by counting squares. (There are 45 full squares and 10 half-squares for a total of 50 of them.) An easier method would be to use knowledge of geometry to calculate the area of that triangle, which is also, by the way, the "area under the curve". Putting the correct values into the formula $A = \frac{1}{2}bh$ we get

$$\text{area of triangle} = \text{area under the curve} = \frac{1}{2} * 10 * 10 = 50.$$

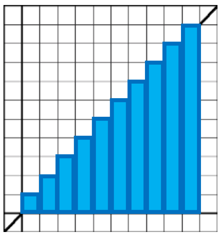
This scenario is fairly easy, because the function $f(x) = x$ is a line, and the "area under the curve" of f forms a well-known geometric shape. What happens when the curve is not linear but actually curves? Since there is no geometric formula for irregularly-shaped spaces, we'll need a way to approximate the area under the curve.



Suppose that we form a series of rectangles under $f(x) = x$ on the interval [0, 10] and use those to approximate the area under the curve. If we draw in 10 rectangles of width = 1, and put the left top corner of each rectangle on the line $f(x) = x$, the sum of the areas of the rectangles = approximation of the area under the curve =

$$0(1) + 1(1) + 2(1) + 3(1) + 4(1) + 5(1) + 6(1) + 7(1) + 8(1) + 9(1) = 45.$$

This value is, of course, lower than the actual area under the curve. Our approximation leaves out the white triangle-shaped space above each rectangle.



Suppose that we now form a series of rectangles under $f(x) = x$ on the interval [0, 10], drawing in 10 rectangles of width = 1, and put the right top corner of each rectangle on the line $f(x) = x$. In this case, the sum of the areas of the rectangles = approximation of the area under the curve =

$$1(1) + 2(1) + 3(1) + 4(1) + 5(1) + 6(1) + 7(1) + 8(1) + 9(1) + 10(1) = 55.$$

This value is, of course, higher than the actual area under the curve. Our approximation includes triangle-shaped space at the top of each rectangle above the graph of f.

Before we continue on, we need some formal vocabulary.

Definition 5.1: "A **partition** of $[a, b]$ is a finite set P of points $x_0, x_1, \ldots, x_n$ such that $a = x_0 < x_1 < \ldots < x_n = b$. We describe P by writing $P = \{x_0, x_1, \ldots, x_n\}$."

In the triangle example above, we used $P = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

For the first approximation above, we inscribed each rectangle so that it lay completely inside the triangle, and noted that our approximation was necessarily lower than the actual triangle area. Our approximation gave us the **lower sum** of $f(x) = x$ associated with the partition P.

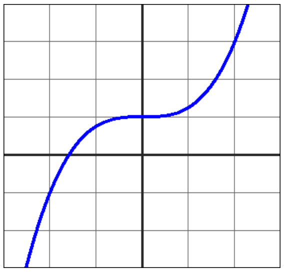
Formally, area of a region R must be at least as large as the lower sum $L_f(P) = m_1 \Delta x_1 + m_2 \Delta x_2 + \ldots + m_n \Delta x_n$. That is, $L_f(P) \le$ area of R.

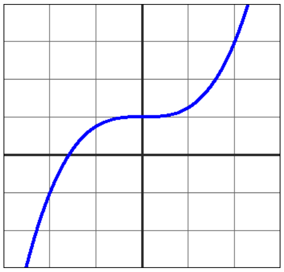
For the second approximation, we circumscribed each rectangle so that all parts of the triangle lay completely inside one of the rectangles, and noted that our approximation was necessarily higher than the actual triangle area. Our approximation gave us the **upper sum** of $f(x) = x$ associated with the partition P.

Formally, the area of a region R can be no larger than the upper sum $U_f(P) = M_1 \Delta x_1 + M_2 \Delta x_2 + \ldots + M_n \Delta x_n$. That is, area of $R \le U_f(P)$.

Example A: Given $f(x)=\frac{1}{4}x^3+1$ and $P=\left\{-2,\ -1,\ 0,\ \frac{1}{2},\ 1,\ 2\right\}$, compute $L_f(P)$ and $U_f(P)$.

answers: $\frac{209}{64}$, $\frac{377}{64}$





Note carefully: The lower sum computed in Example A does not correspond directly to a geometric concept of area. (In computing the lower sum, we found a rectangle with a "negative area".)

If we want to equate our rectangle-based sums to geometric area, we'd need all non-negative *y*-coordinates within our partition, as was the case for the upper sum in Example A. In general, for a region *R* in-between the horizontal axis and the graph of a *continuous non-negative* function *f*, it will always be true that

$$L_f(P) \le \text{area of } R \le U_f(P).$$

Foreshadowing: A "negative area under a curve" will have a meaning in calculus. We'll explore this later.

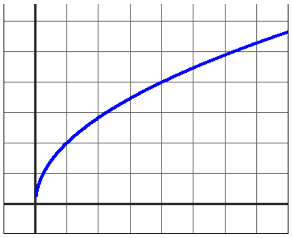
Definition 5.2: "Let *f* be continuous on [*a*, *b*], and let $P=\{x_0, x_1, \ldots, x_n\}$ be any partition of [*a*, *b*]. For each *k* between 1 and *n*, let t_k be an arbitrary number in $[x_{k-1}, x_k]$. Then the sum

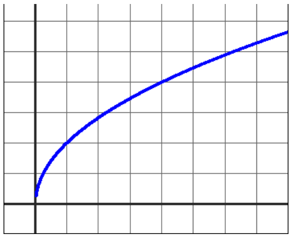
$$f(t_1)\Delta x_1 + f(t_2)\Delta x_2 + \ \ldots\ + f(t_n)\Delta x_n$$

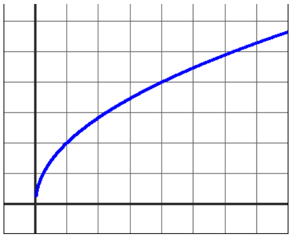
is called a **Riemann sum** for *f* on [*a*, *b*] and is denoted $\sum_{k=1}^{n} f(t_k)\Delta x_k$."

Three choices for t_k that we'll explore in the next Example are left endpoints, midpoints, and right endpoints of each subinterval $[x_{k-1}, x_k]$. The resulting Riemann sums will be called left sum, midpoint sum and right sum.

Example B: Approximate the area under the curve $y=2\sqrt{x}$ on the interval [2, 7] using a partition having 5 subintervals of the same length, and left sum, midpoint sum, and right sum.
answers: ≈ 19.66364418, ≈ 20.93601101, ≈ 22.12671968



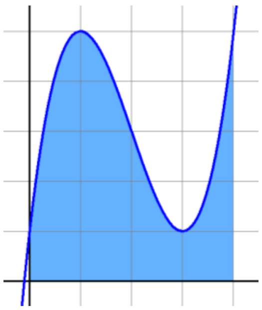


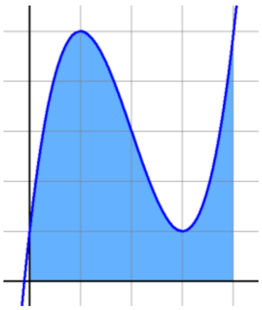


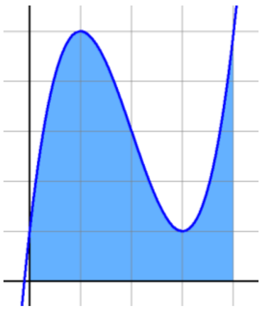
Example B extended: Repeat the approximation process using a partition with 10 subintervals of the same length (left sum, midpoint sum, and right sum). *answers*: ≈ 20.29982759, ≈ 20.92586181, ≈ 21.53136534

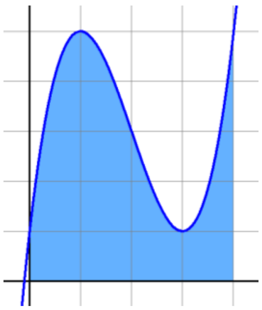
The exact value for the area under the curve $y = 2\sqrt{x}$ on the interval $2 \le x \le 7$ is $\frac{4}{3}\left(\sqrt{7^3} - \sqrt{2^3}\right)$ which is approximately 20.92244274. Just as increasing the number of subintervals in our partition brought us closer to the true value for the area under the curve $y = 2\sqrt{x}$, it is reasonable to suppose that, in general, for any function, increasing the number of subintervals will provide an increasingly better approximation to the area under the curve. In section 5.2 we'll look at the limit of a Riemann sum as the number of subintervals n approaches ∞.

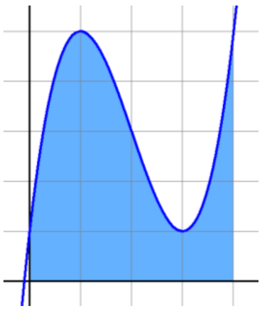
Practice in identifying x-values and $f(x)$ values for various sums; $P = \{\ 0,\ 1,\ \ 2,\ 3,\ 4\ \}$











for $L_f(P)$:

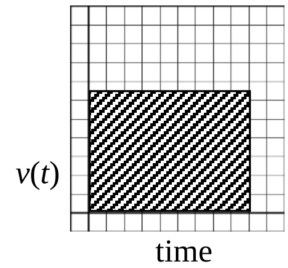
for $U_f(P)$:

for left sum:

for midpoint sum:

for right sum:

Now comes an important question: Why would we be interested in the area under a curve?
Consider a velocity function $v(t)$. When $v(t)$ is constant, it is not difficult to see that the formula "distance = rate of speed $*$ time" is the area of the rectangle formed on the graph, i.e. "distance = area under the curve $v(t)$".



When $v(t)$ is changing, the area of each rectangle formed by our subintervals gives us "average rate of speed on the subinterval $*$ time = area under the curve = distance".

We've already determined that "velocity = rate of change of distance = first derivative of distance".
This relationship turned backwards is "distance = antiderivative of velocity".
Substituting from the observations above we will conclude

"area under the curve $v(t)$ = antiderivative of $v(t)$ = integral of $v(t)$".

A similar thought process can be applied to other functions we've encountered.